

15.2 Double Integrals over General Region

Last time:

For the rectangular region, R :

$$a \leq x \leq b, \quad c \leq y \leq d$$

we learned

$$\iint_R f(x, y) dA$$

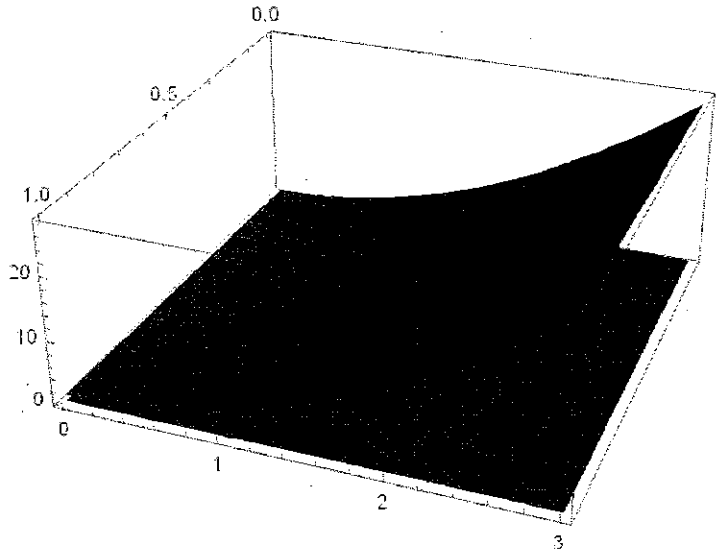
$$= \int_a^b \left(\int_c^d f(x, y) dy \right) dx$$

$$= \int_c^d \left(\int_a^b f(x, y) dx \right) dy$$

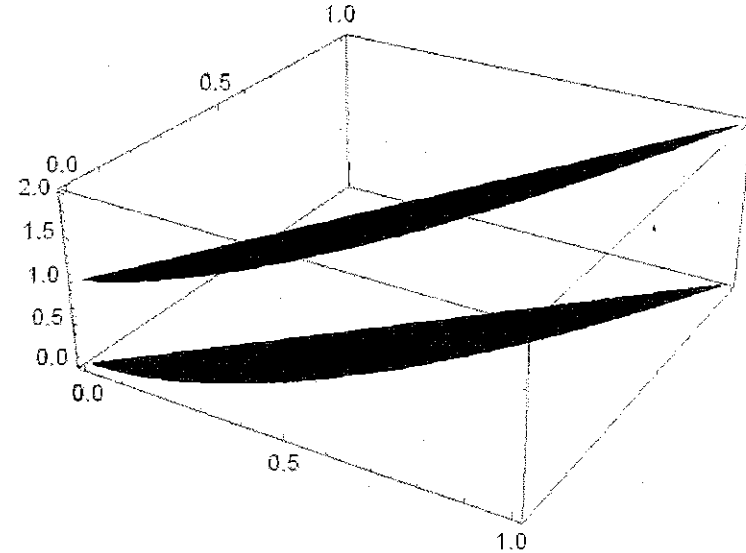
Type 1 (Top/Bot)	Type 2 (Left/Right)
<p>For all x in the range, $a \leq x \leq b$, we have $g_1(x) \leq y \leq g_2(x)$</p>	<p>For all y in the range, $c \leq y \leq d$, we have $h_1(y) \leq x \leq h_2(y)$</p>
$\int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x, y) dy \right) dx$	$\int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x, y) dx \right) dy$

In 15.2, we discuss regions other than rectangles.

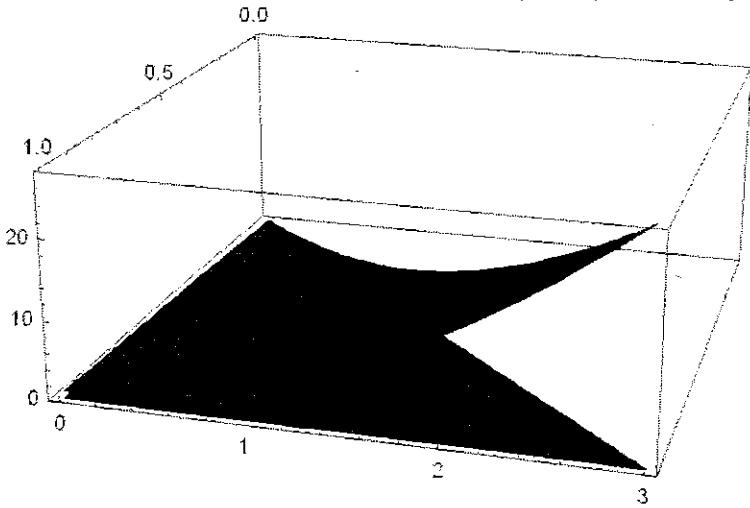
The surface $z = x + 3y^2$ over the rectangular region $R = [0,1] \times [0,3]$



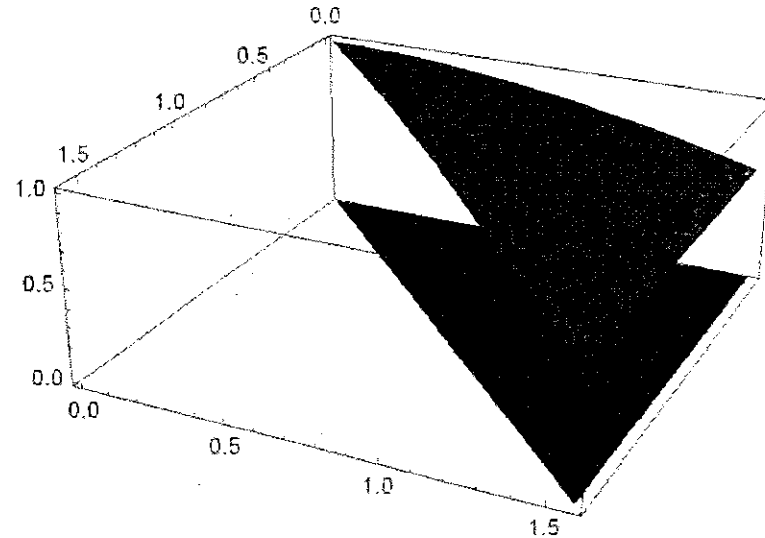
The surface $z = x + 1$ over the region bounded by $y = x$ and $y = x^2$.



The surface $z = x + 3y^2$ over the triangular region with corners $(0,0)$, $(1,0)$, and $(1,3)$.



The surface $z = \sin(y)/y$ over the triangular region with corners $(0,0)$, $(0, \pi/2)$, $(\pi/2, \pi/2)$.



Examples:

1. Let D be the triangular region in the xy -plane with corners $(0,0)$, $(1,0)$, $(1,3)$.

Evaluate $\iint_D x + 3y^2 dA$

OPTION 1: FIX x FIRST

$$0 \leq x \leq 1$$

$$\Rightarrow 0 \leq y \leq 3x$$

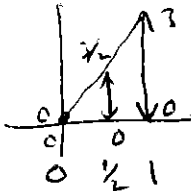
$$\int_0^1 \left(\int_0^{3x} x + 3y^2 dy \right) dx$$

$$\int_0^1 xy + y^3 \Big|_0^{3x} dx$$

$$\int_0^1 3x^2 + 27x^3 dx$$

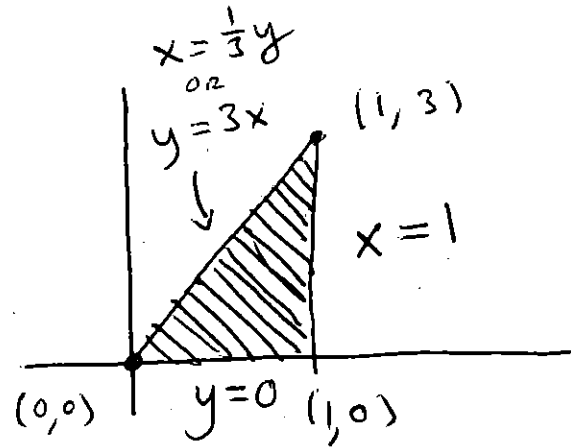
$$= x^3 + \frac{27}{4} x^4 \Big|_0^1$$

$$= 1 + \frac{27}{4} = \boxed{\frac{31}{4}} = 7.75$$



Ex)

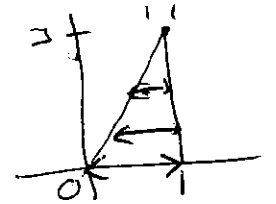
x	
0	$0 \leq y \leq 0$
1	$0 \leq y \leq 3$
$\frac{1}{2}$	$0 \leq y \leq \frac{3}{2}$



OPTION 2: FIX y FIRST

$$0 \leq y \leq 3$$

$$\Rightarrow \frac{1}{3} y \leq x \leq 1$$



Ex)

y	
0	$0 \leq x \leq 1$
3	$1 \leq x \leq 1$
1	$\frac{1}{3} \leq x \leq 1$
2	$\frac{2}{3} \leq x \leq 1$

$$\int_0^3 \left(\int_{\frac{1}{3}y}^1 x + 3y^2 dx \right) dy$$

$$= \int_0^3 \left. \frac{1}{2} x^2 + 3y^2 x \right|_{\frac{1}{3}y}^1 dy$$

$$= \int_0^3 \left(\frac{1}{2} + 3y^2 - \frac{1}{18} y^2 - y^3 \right) dy$$

$$= \left. \frac{1}{2} y + y^3 - \frac{1}{54} y^3 - \frac{1}{4} y^4 \right|_0^3$$

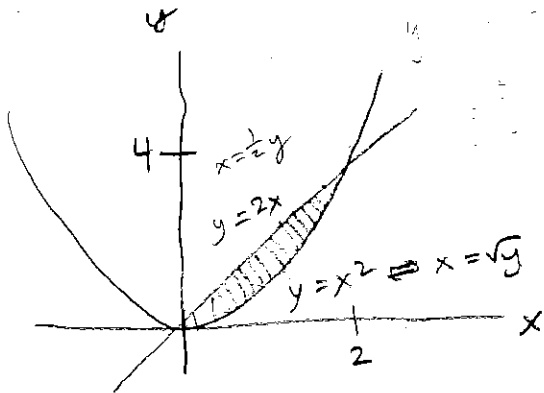
$$= \frac{3}{2} + 27 - \frac{1}{2} - \frac{81}{4} = 1 + \frac{108}{4} - \frac{81}{4}$$

$$= 1 + \frac{27}{4} = \boxed{\frac{31}{4}} = 7.75$$

2. Find the volume of the solid bounded by the surfaces $z = x + 1$, $y = x^2$, $y = 2x$, $z = 0$.

Step 1 $\iint_D x+1 \, dA$

STEP 2



$$x^2 = 2x \Rightarrow x^2 - 2x = 0 \Rightarrow x(x-2) = 0$$

$x=0$ or $x=2$

OPTION 1: $0 \leq x \leq 2$

$$\Rightarrow x^2 \leq y \leq 2x$$

$$\begin{aligned} & \int_0^2 \left(\int_{x^2}^{2x} x+1 \, dy \right) dx \\ &= \int_0^2 (xy + y) \Big|_{x^2}^{2x} dx \\ &= \int_0^2 (2x^2 + 2x - x^3 - x^2) dx \\ &= \left[\frac{1}{3}x^3 + x^2 - \frac{1}{4}x^4 \right]_0^2 \\ &= \frac{8}{3} + 4 - 4 = \boxed{\frac{8}{3}} \end{aligned}$$

OPTION 2: $0 \leq y \leq 4$

$$\frac{1}{2}y \leq x \leq \sqrt{y}$$

$$\begin{aligned} &= \int_0^4 \left(\int_{\frac{1}{2}y}^{\sqrt{y}} x+1 \, dx \right) dy \\ &= \int_0^4 \left(\frac{1}{2}x^2 + x \right) \Big|_{\frac{1}{2}y}^{\sqrt{y}} dy \\ &= \int_0^4 \left(\frac{1}{2}\sqrt{y} + \sqrt{y} - \frac{1}{8}y^2 - \frac{1}{2}y \right) dy \\ &= \left[\frac{2}{3}y^{3/2} - \frac{1}{24}y^3 \right]_0^4 \\ &= \frac{16}{3} - \frac{1}{3} = \boxed{\frac{15}{3}} = 5 \end{aligned}$$

≈ 2.6

3. Draw the region of integration for

$$\int_0^{\pi/2} \int_x^{\pi/2} \frac{\sin(y)}{y} dy dx$$

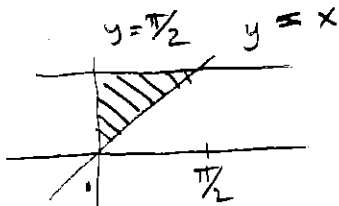
then switch the order of integration.

GIVEN $0 \leq x \leq \frac{\pi}{2}$
 $\Rightarrow x \leq y \leq \frac{\pi}{2}$

STEP 1: TOP/BOT or LEFT/RIGHT?

BOT $\rightarrow y = x$

TOP $\rightarrow y = \frac{\pi}{2}$



STEP 2: TICKMARKS $x=0$ to $x=\frac{\pi}{2}$

REVERSE ORDER

$$0 \leq y \leq \frac{\pi}{2}$$

$$\Rightarrow 0 \leq x \leq y$$

$$\int_0^{\pi/2} \left(\int_0^y \frac{\sin(y)}{y} dx \right) dy$$

$$\int_0^{\pi/2} \frac{\sin(y)}{y} x \Big|_0^y dy$$

$$\int_0^{\pi/2} \frac{\sin(y)}{y} \cdot y - 0 dy$$

$$\int_0^{\pi/2} \sin(y) dy$$

$$-\cos(y) \Big|_0^{\pi/2}$$

$$-\underbrace{\cos(\frac{\pi}{2})}_0 - -\underbrace{\cos(0)}_1$$

$$= \boxed{1}$$

YOU MUST DRAW THE PICTURE!!!

THAT IS THE BEST WAY TO DO IT!

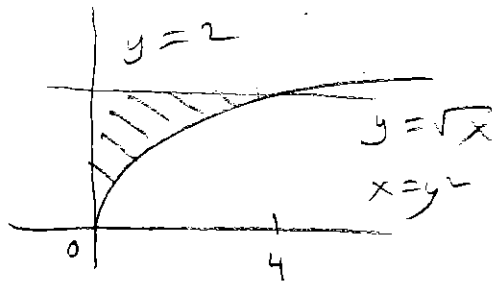
4. Switch the order of integration for

$$\int_0^4 \int_{\sqrt{x}}^2 \sin(y^3) dy dx$$

GIVEN $0 \leq x \leq 4$
 $\Rightarrow \sqrt{x} \leq y \leq 2$

BOT $\rightarrow y = \sqrt{x}$

TOP $\rightarrow y = 2$



$$0 \leq y \leq 2$$

$$0 \leq x \leq y^2$$

$$\int_0^2 \int_0^{y^2} \sin(y^3) dx dy$$

$$\int_0^2 \sin(y^3) x \Big|_0^{y^2} dy$$

$$\int_0^2 y^2 \sin(y^3) - 0 dy$$

$$\int_0^8 y^2 \sin(u) \frac{1}{3y^2} du$$

$$u = y^3$$

$$du = 3y^2 dy$$

$$\frac{1}{3y^2} du = dy$$

$$\frac{1}{3} \int_0^8 \sin(u) du$$

$$-\frac{1}{3} \cos(u) \Big|_0^8$$

$$-\frac{1}{3} \cos(8) - -\frac{1}{3} \cos(0)$$

$$\boxed{-\frac{1}{3} \cos(8) + \frac{1}{3}} \approx 0.231873$$

Setting up a problem given in "words":

1. Find integrand

Solve for "z" anywhere you see it.

If there are two z's, then set up two double integrals (subtract at end).

2. Region?

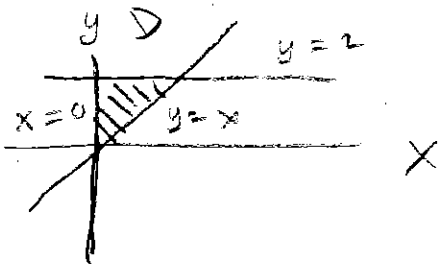
Graph the region in the xy-plane.

- Graph all given x and y constraints.
- And find the xy-curves where the surfaces (the z's) intersect.

Examples (directly from HW):

HW 15.2: Find the volume enclosed by $z = 4x^2 + 4y^2$ and the planes $x = 0$, $y = 2$, $y = x$, and $z = 0$.

$$\iint_D 4x^2 + 4y^2 dA$$



HW 15.3:

Find the volume below $z = 18 - 2x^2 - 2y^2$ and above the xy-plane. $\leftarrow z=0$

$$\iint_D 18 - 2x^2 - 2y^2 dy$$

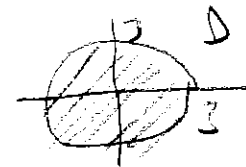
No xy-constraints given???

INTERSECT!

$$18 - 2x^2 - 2y^2 = 0$$

$$\Rightarrow 18 = 2x^2 + 2y^2$$

$$\Rightarrow 9 = x^2 + y^2$$



HW 15.3:

Find the volume enclosed by
 $-x^2 - y^2 + z^2 = 22$ and $z = 5$.

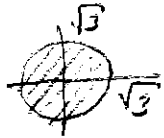
$$z = \pm \sqrt{22 + x^2 + y^2}$$



$$\iint_D 5 \, dA \quad \text{AND} \quad \iint_D \sqrt{22 + x^2 + y^2} \, dA$$

INTERSECT

$$\begin{aligned} -x^2 - y^2 + 25 &= 22 \\ 3 &= x^2 + y^2 \end{aligned}$$

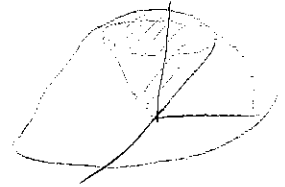


HW 15.3:

Find the volume above the upper cone

$z = \sqrt{x^2 + y^2}$ and
 below $x^2 + y^2 + z^2 = 81$

$$z = \pm \sqrt{81 - x^2 - y^2}$$



$$\iint_D \sqrt{81 - x^2 - y^2} \, dA \quad \text{AND} \quad \iint_D \sqrt{x^2 + y^2} \, dA$$

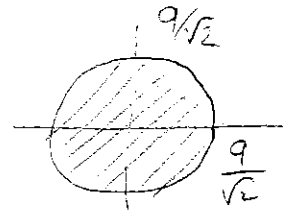
INTERSECT

$$x^2 + y^2 + (\sqrt{x^2 + y^2})^2 = 81$$

$$\Rightarrow x^2 + y^2 + x^2 + y^2 = 81$$

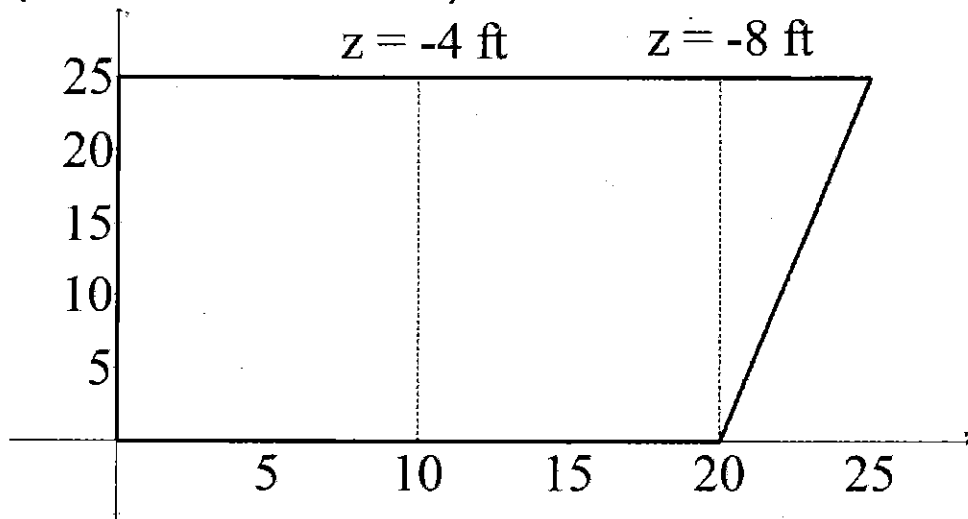
$$\Rightarrow 2x^2 + 2y^2 = 81$$

$$\Rightarrow x^2 + y^2 = \frac{81}{2}$$



An applied problem:

Your swimming pool has the following shape (viewed from above)



The bottom of the pool is a plane with depths as indicated (the pool gets deeper in a linear way from left-to-right)

Solution:

1. Surface?

Slope in y-direction = 0

Slope in x-direction = $-4/10 = -0.4$

Also the plane goes through $(0, 0, 0)$

Thus, the plane that describes the bottom of the pool is: $z = -0.4x + 0y$

2. Region?

The line on the right goes through $(20,0)$ and $(25,25)$, so it has slope = 5 and it is given by the equation

$$y = 5(x-20) = 5x - 100$$

or $x = (y+100)/5 = 1/5 y + 20$

The best way to describe this region is by thinking of it as a left-right region.

On the left, we have $x = 0$

On the right, we have $x = 1/5 y + 20$

Therefore, we have

$$\int_0^{25} \left(\int_0^{\frac{1}{5}y+20} -0.4x \, dx \right) dy = -741.\bar{6} \text{ ft}^3$$